Given a set of connectivity graphs \( \{W_s\} \) with positive weights \( w_{ij} > 0 \), Graph Laplacian (positive semi-definite matrix) and Regularized Graph Laplacian (symmetric positive definite matrix, SPD) can be computed as follows:

\[
L_s = D_s - W_s
\]

\[
D_s = \text{diag}\left( \sum_j w_{ij} \right)
\]

\[
\hat{L}_s = L_s + \gamma I
\]

\( \gamma > 0 \)

Euclidean distance between SPD matrices is suboptimal. It is better to consider the Geodesic Distance on the Riemannian manifold.

**1- Symmetric Positive Definite Graph Representation**

- **Affine Invariant Metric:**
  \[
d_{af}(\hat{L}_1, \hat{L}_2) = \| \text{log}(\hat{L}_1^{\frac{1}{2}} \hat{L}_2^{\frac{1}{2}}) \|_F
\]

- **Log Euclidean Distance** [1]
  \[
d_{\text{log}E}(\hat{L}_1, \hat{L}_2) = \| \log(\hat{L}_1) - \log(\hat{L}_2) \|_F
\]

- **Stein Divergence** [2]
  \[
d_{\text{stein}}(\hat{L}_1, \hat{L}_2) = \left( \log \det \left( \frac{\hat{L}_1 + \hat{L}_2}{2} \right) - \frac{\log \det(\hat{L}_1 \hat{L}_2)}{2} \right)^{1/2}
\]

- **Log Euclidean Gaussian Kernel (GK-LogE)**
  \[
  K_{\text{log}E}(i, j) = \exp \left( -\frac{d_{\text{log}E}(\hat{L}_i, \hat{L}_j)^2}{\sigma^2} \right)
  \]

- **Stein Gaussian Kernel (GK-Stein)**
  \[
  K_{\text{stein}}(i, j) = \exp(-\beta d_{\text{stein}}(\hat{L}_i, \hat{L}_j))
  \]

- **Log Euclidean Linear Kernel (LK-LogE)**
  \[
  K = (d_{\text{log}E})^T \times d_{\text{log}E}
  \]

**2- Riemannian Manifold of SPD Matrices**

**Alternative Metrics on the Manifold Computationally Faster**

- **Log Euclidean Distance** [1]
  \[
d_{\text{log}E}(\hat{L}_1, \hat{L}_2) = \| \log(\hat{L}_1) - \log(\hat{L}_2) \|_F
\]

**Riemannian Metric, Geodesic Distance, Easy to Kernelize**

**Stein Divergence** [2]

**Not a Riemannian Metric, based on convex structure of manifold**

**3- Riemannian Kernel**

**Positive definite with**

\[
\beta \in \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, \frac{n-1}{2} \right\} \cup \left\{ \tau \in \mathbb{R} : \tau > \frac{n-1}{2} \right\}
\]

**4- Results**

Leave-One-Out with Riemannian-Kernel SVM

Parameters fitted with Cross-Validation

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Riemannian-Kernel SVM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Autism</strong></td>
<td><strong>Autism</strong></td>
</tr>
<tr>
<td>DTI</td>
<td>68.00 %</td>
</tr>
<tr>
<td>LK-LogE</td>
<td>59.27 %</td>
</tr>
<tr>
<td>LK-Stein</td>
<td>54.26 %</td>
</tr>
<tr>
<td>L-SVM</td>
<td>55.32 %</td>
</tr>
<tr>
<td><strong>Euclidean</strong></td>
<td><strong>G-SVM</strong></td>
</tr>
<tr>
<td>G-SVM</td>
<td>68.00 %</td>
</tr>
<tr>
<td>L-SVM</td>
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<td><strong>GK-LogE</strong></td>
<td>54.26 %</td>
</tr>
<tr>
<td><strong>GK-Stein</strong></td>
<td>54.26 %</td>
</tr>
</tbody>
</table>

**Higher performances on Riemannian manifold vs. Euclidean space**

**Best results with LogE distance (Geodesic) compared to Stein divergence**

**LogE and Stein (for certain \( \beta \)) define valid Gaussian kernels**

**LogE and Stein are faster to compute than Affine Invariant Metric**

**Results independent of \( \gamma \)**

**References**

1. V. Arsigny et al., *Fast and simple calculus on tensors in the Log-Euclidean framework*, MICCAI, 2009